MATH 147 QUIZ 11 SOLUTIONS

1. Verify the Divergence Theorem for $\mathbf{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$, and B the solid rectangle $0 \le x \le a$, $0 \le y \le b$, and $0 \le z \le c$. (5 points)

Recall that the divergence theorem states, for a closed surface S bounding the solid B,

$$\iint_S \mathbf{F} \cdot dS = \iiint_B \text{div}\mathbf{F}\, dV.$$

To verify the theorem, we must compute both integrals and see that they are the same.
For the Surface integral, we note that the surface we must integrate over is actually the combination of the 6 faces of the rectangle. Each of these is relatively straightforward to integrate: The normal vector to the bottom face would be $-\vec{k}$, so the surface integral would be

$$\iint_S F \cdot (-\vec{k})\, dS = \iint_S -z^2\, dS = 0.$$

This is because $z = 0$ on the bottom face. On the other hand, the top face has normal vector $\vec{k}$, and so the integral is

$$\iint_S F \cdot \vec{k}\, dS = \iint_S z^2\, dS = abc^2.$$

We do something similar for all of these, and the sum of the 6 integrals should be $abc(a + b + c)$. Note one could also parameterize the bottom face as $S(x, y) = x\vec{i} + y\vec{j} + 0\vec{k}$ and get the same result.
On the other hand, to calculate the right hand integral, we first find the divergence of F, which is $2(x + y + z)$. Then, we have the integral

$$\int_0^c \int_0^b \int_0^a 2x + 2y + 2z\, dx\, dy\, dz = \int_0^c \int_0^b a^2 + 2ay + 2az\, dy\, dz = \int_0^c a^2b + ab^2 + 2abz\, dz$$
$$= a^2bc + ab^2c + abc^2 = abc(a + b + c).$$

And so, the theorem is verified for this case.

2. Use a line integral to find the area contained in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. (5 points)

We know that we can convert between line integrals and surface integrals using Green's Theorem. To find the area using the line integral, we need a function whose curl is 1. Then, Green's theorem would state

$$\int_{\partial D} \mathbf{F} \cdot dr = \iint_D \text{Curl } \mathbf{F}\, dA = \iint_D 1\, dA = \text{Area of D}.$$

The typical function to use, and the one discussed in class is $\mathbf{F} = \frac{-y}{2}\vec{i} + \frac{x}{2}\vec{j}$. It is straightforward to verify that Curl $\mathbf{F} = 1$. Now, we compute the line integral over the boundary given by the ellipse. We parameterize the ellipse in the standard way (or by doing a polar substitution followed by a linear scaling) to get $r(t) = a\cos(t)\vec{i} + b\sin(t)\vec{j}$. One can verify that as t goes from 0 to 2π, we remain on the ellipse and traverse all the way around. Next, $r'(t) = -a\sin(t)\vec{i} + b\cos(t)\vec{j}$. The last piece is $\mathbf{F}(r(t)) = (-b\sin(t)/2)\vec{i} + (a\cos(t)/2)\vec{j}$. We can now compute the integral as

$$\int_0^{2\pi} \mathbf{F} \cdot r'(t)\, dt = \int_0^{2\pi} ab\sin^2(t)/2 + ab\cos^2(t)/2\, dt = ab/2 \int_0^{2pi} dt = ab\pi.$$